ON MAXIMAL SUBALGEBRAS AND A GENERALISED JORDAN-HÖLDER THEOREM FOR LIE ALGEBRAS

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Abstract

The purpose of this paper is to continue the study of chief factors of a Lie algebra and to prove a further strengthening of the Jordan-Hölder Theorem for chief series.
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1 Introduction

Throughout L will denote a finite-dimensional Lie algebra over a field F. The factor algebra A/B is called a *chief factor* of L if B is an ideal of L and A/B is a minimal ideal of L/B. A chief factor A/B is called *Frattini* if $A/B \subseteq \phi(L/B)$. This concept was first introduced in [3].

If there is a subalgebra, M such that $L = A + M$ and $B \subseteq A \cap M$, we say that A/B is a *supplemented* chief factor of L, and that M is a *supplement* of A/B in L. Also, if A/B is a non-Frattini chief factor of L, then A/B is *supplemented* by a maximal subalgebra M of L.

If A/B is a chief factor of L supplemented by a subalgebra M of L, and $A \cap M = B$ then we say that A/B is *complemented* chief factor of L, and M is a *complement* of A/B in L. When L is *solvable*, it is easy to see that a chief factor is Frattini if and only if it is not complemented.

If U is a subalgebra of L, the *core* of U, U_L, is the largest ideal of L contained in U. We say that U is core-free in L if $U_L = 0$.

We shall call L *primitive* if it has a core-free maximal subalgebra. Then we have the following characterisation of primitive Lie algebras.

Theorem 1.1 *([4, Theorem 1.1])*

(i) A Lie algebra L is primitive if and only if there exists a subalgebra M of L such that $L = M + A$ for all minimal ideals A of L.

(ii) Let L be a primitive Lie algebra. Assume that U is a core-free maximal subalgebra of L and that A is a non-trivial ideal of L. Write $C = C_L(A)$. Then $C \cap U = 0$. Moreover, either $C = 0$ or C is a minimal ideal of L.

(iii) If L is a primitive Lie algebra and U is a core-free maximal subalgebra of L, then exactly one of the following statements holds:

- *(a) $Soc(L) = A$ is a self-centralising abelian minimal ideal of L which is complemented by U; that is, $L = U \dot{+} A$.*
- *(b) $Soc(L) = A$ is a non-abelian minimal ideal of L which is supplemented by U; that is $L = U + A$. In this case $C_L(A) = 0$.*
- *(c) $Soc(L) = A \oplus B$, where A and B are the two unique minimal ideals of L and both are complemented by U; that is, $L = A \dot{+} U = B \dot{+} U$. In this case $A = C_L(B)$, $B = C_L(A)$, and A, B and $(A + B) \cap U$ are nonabelian isomorphic algebras.*

We say that L is

- *primitive of type* 1 if it has a unique minimal ideal that is abelian;
- *primitive of type* 2 if it has a unique minimal ideal that is non-abelian; and
- *primitive of type* 3 if it has precisely two distinct minimal ideals each of which is non-abelian.

If A/B be a supplemented chief factor of L for which M is a maximal subalgebra of L supplementing A/B in L such that L/M_L is monolithic and primitive, we call M a *monolithic maximal subalgebra* supplementing A/B in L. Note that [4, Proposition 2.5 (iii) and (iv)] show that such an M exists.

We say that two chief factors are *L-isomorphic*, denoted by '$\cong_L$', if they are isomorphic both as algebras and as L-modules. Note that if L is a primitive Lie algebra of type 3, its two minimal ideals are not L-isomorphic, so we introduce the following concept. We say that two chief factors of L are *L-connected* if either they are L-isomorphic, or there exists an epimorphic image $\overline{L}$ of L which is primitive of type 3 and whose minimal ideals are L-isomorphic, respectively, to the given factors. (It is clear that, if two chief factors of L are L-connected and are not L-isomorphic, then they are nonabelian and there is a single epimorphic image of L which is primitive of type 3 and which connects them.)

Our primary objective is to generalise further the version of the Jordan-Hölder Theorem for chief series of L established in [4]. Our result could probably be obtained from [2], but we prefer to follow the approach adopted for groups (though in a more general context) in [1] as interesting results and concepts are obtained on the way.

2 Preliminary results on chief factors

Let A/B and C/D be chief factors of L. We write $A/B \searrow C/D$ if $A = B+C$ and $B \cap C = D$. Clearly, if A/B is abelian, then so is C/D.

Lemma 2.1 *Let $A/B \searrow C/D$. Then*

(i) if A/B is supplemented by M in L, then so is C/D;

(ii) if M supplements A/B in L and K supplements B/D in L, then $C+M\cap K$ supplements A/C in L and, in this case, $M\cap K$ supplements A/D in L; and

(iii) (i) and (ii) both hold with 'supplemented' replaced by 'complemented'.

If, further, C/D is non-abelian, then

(iv) the set of monolithic supplements of C/D in L coincides with the set of monolithic supplements of A/B in L;

(v) *if B/D is an abelian chief factor of L then the (possibly empty) set of complements of B/D in L coincides with the set of complements of A/C in L.*

Proof. We have $A = B + C$ and $B \cap C = D$.

(i) Suppose that $L = A + M$ and $B \subseteq A \cap M$. Then $L = B + C + M = C + M$ and $D = B \cap C \subseteq A \cap M \cap C = M \cap C$.

(ii) Suppose that $L = A+M = B+K$, $B \subseteq A\cap M$ and $D \subseteq B\cap K$. Then $A+C+M\cap K = A+M\cap K = B+C+M\cap K = C+M\cap(K+B) = C+M = L$. Furthermore, $C = C+D \subseteq C+B\cap K \subseteq C+(A\cap M\cap K) = A\cap(C+M\cap K)$, so $C+M\cap K$ supplements A/C in L.

Moreover, in this case, $L = A+M\cap K$ and $D \subseteq B\cap K \subseteq A\cap M\cap K$, so $M\cap K$ supplements A/D in L.

(iii) Simply substituting equalities for the inequalities in the above proofs will yield the corresponding results for complements.

For the remaining parts we can assume, without loss of generality that $D = 0$ and that C is a non-abelian minimal ideal of L. It follows that $[C, B] \subseteq C \cap B = 0$.

(iv) If M is a monolithic supplement of C in L then $L = C+M$ and L/M_L is a primitive Lie algebra of type 2. Now $\mathrm{Soc}(L/M_L) = (C+M_L)/M_L$ and $C_L(C) = C_L((C+M_L)/M_L) = M_L$. Hence $B \subseteq C_L(C) = M_L \subseteq M$. Thus $L = A+M$ with $B \subseteq A\cap M$ and M is a monolithic supplement of A/B in L.

Conversely, if M is a monolithic supplement of A/B in L, then M supplements C in L, by (i).

(v) Suppose that B is an abelian ideal of L complemented by M, so $L = B\dot{+}M$. Then $C_L(B) = B \oplus M_L$ and $A = B \oplus A\cap M_L$. Since C is non-abelian, this implies that $C = C^2 = A^2 \subseteq A\cap M_L$. Thus $C \subseteq M$ and M complements A/C in L.

Conversely, suppose that $L = A+M$ and $A\cap M = C$. Then $L = B+C+M = B+M$ and $C = A\cap M = C+B\cap M$, so $B\cap M \subseteq B\cap C = 0$. Hence M complements B in L.

□

Lemma 2.2 *Let U and S be two maximal subalgebras of a Lie algebra L such that $U_L \neq S_L$. Suppose that U and S supplement the same chief factor A/B of L. Then $M = A + U \cap S$ is a maximal subalgebra of L such that $M_L = A + U_L \cap S_L$.*

(i) Assume that A/B is abelian. Then M is a maximal subalgebra of type 1 which complements the chief factors $U_L/U_L \cap S_L$ and $S_L/U_L \cap S_L$. Moreover, $M \cap U = M \cap S = U \cap S$.

(ii) Assume that A/B is non-abelian. Then either U or S is of type 3. Suppose that U is of type 3 and S is monolithic. Then $U_L \subset S_L = C_L(A/B)$. Moreover, M is a maximal subalgebra of type 2 of L which supplements the chief factor S_L/U_L.

(iii) Assume that U and S are of type 3. Then M is a maximal subalgebra of type 3 of L which complements the chief factors $(A + S_L)/M_L$ and $(A + U_L)/M_L$. Moreover $M \cap U = M \cap S = U \cap S$.

Proof. We have that $L = A + U = A + S$ and $B \subseteq A \cap U \cap S$.

(i) Let A/B be abelian and put $C = C_L(A/B)$. First note that $B \subseteq A \cap U \subset A$ and $A \cap U$ is an ideal of L since A/B is abelian, so $B = A \cap U$. Thus $M \cap U = (A + U \cap S) \cap U = A \cap U + U \cap S = B + U \cap S = U \cap S$. Similarly $M \cap S = U \cap S$. Since $U \neq S$, M is a proper subalgebra of L.

Clearly $C = A + U_L = A + S_L$. But $B \subseteq A \cap (U_L + S_L) \subseteq A$, so $A \cap (U_L + S_L) = B$ or A. The former implies that $U_L + S_L = U_L + A \cap (U_L + S_L) = U_L + B = U_L$ and $U_L + S_L = S_L$ similarly, contradicting the fact that $U_L \neq S_L$. Hence the latter holds and $C = U_L + S_L$. But now $U_L/U_L \cap S_L \cong_L C/S_L \cong_L A/B$ since $A \cap S_L = B$.

We have $L = S + A = S + C = S + U_L + S_L = S + U_L$, so $M + U_L = A + U \cap S + U_L = A + U \cap (S + U_L) = A + U = L$. Moreover, $U_L \cap S_L \subseteq M \cap U_L$ so M is a maximal subalgebra of L which complements the abelian chief factor $U_L/U_L \cap S_L$. Similarly M complements $S_L/U_L \cap S_L$.

Finally $L = M + U_L = M + C$ and $M \cap C = M \cap (A + U_L) = A + M \cap U_L = A + U_L \cap S_L$, so M also complements $C/(A + U_L \cap S_L)$, whence $M_L = A + U_L \cap S_L$.

(ii) Assume that A/B is non-abelian. If U and S were both monolithic of type 2, then $U_L = S_L = C_L(A/B)$, contradicting our hypothesis. It follows that either U or S is of type 3.

So assume that U is of type 3 and that S is monolithic. Then $S_L = C_L(A/B)$. Note that $(A+U_L)/U_L$ is a chief factor of L which is L-isomorphic to A/B. Hence $(A+U_L)/U_L$ and S_L/U_L are the two minimal ideals of the primitive Lie algebra L/U_L of type 3. Both of these are complemented by U; in particular, $L = U + S_L$.

Now $M + S_L = A + U \cap S + S_L = A + (U + S_L) \cap S = A + S = L$ and $U_L = U_L + B = U_L + A \cap S_L = (U_L + A) \cap S_L \subseteq (U \cap S + A) \cap S_L = M \cap S_L$. It follows that M supplements the chief factor S_L/U_L in L.

The quotient algebra

$$\frac{L}{A+U_L} = \frac{A+S_L}{A+U_L} + \frac{M}{A+U_L}$$

is primitive of type 2, by [4, Theorem 1.7, part 2]. But then the ideal $M_L/(A+U_L)$ must be trivial, since otherwise we have $A + S_L \subseteq M_L$ which implies that $S_L \subseteq M$, a contradiction. Hence $M_L = A + U_L$.

Let T be a subalgebra of L such that $U \cap S \subseteq T \subseteq U$. Then $S = U \cap S + S_L \subseteq T + S_L \subseteq U + S_L = L$. Since S is a maximal subalgebra of L, we have that $T + S_L = S$ or L. But then, since $T \cap S_L = U \cap S_L = U_L$, we have $U \cap (T + S_L) = T + U \cap S_L = T$, so $T = U \cap S$ or $T = U \cap L = U$. Hence $U \cap S$ is a maximal subalgebra of U. The image of $U \cap S/U \cap A$ under the isomorphism from $U/U \cap A$ onto L/A is M/A, and so M is a maximal subalgebra of L of type 2.

(iii) Assume now that U and S are maximal subalgebras of type 3, so that the quotient algebras L/U_L and L/S_L are primitive Lie algebras of type 3.

If $C = C_L(A/B)$, then U complements the chief factors $(A+U_L)//U_L$ and C/U_L. Similarly, S complements the chief factors $(A+S_L)//S_L$ and C/S_L. In particular, $U_L \not\subseteq S_L$ and $S_L \not\subseteq U_L$. Hence $L = U + S_L = S + U_L$. But now, by a similar argument to that at the end of (ii), we have that $M = A + U \cap S$ is a maximal subalgebra of L.

Now C/S_L and C/U_L are chief factors of L and $U_L \neq S_L$, so $C = U_L + S_L$. Put $H = U_L \cap S_L$. Then

$$\frac{A+U_L}{A+H} \cong_L \frac{U_L}{U_L \cap (A+H)} = \frac{U_L}{H} \cong_L \frac{C}{S_L},$$

and so $(A+U_L)/(A+H)$ is a chief factor of L and

$$C_L\left(\frac{A+U_L}{A+H}\right) = C_L\left(\frac{C}{S_L}\right) = A + S_L.$$

Similarly $(A+S_L)/(A+H)$ is a chief factor of L and

$$C_L\left(\frac{A+S_L}{A+H}\right) = A+U_L.$$

It follows that the quotient Lie algebra $\bar{L} = L/(A+H)$ has two minimal ideals, namely $\bar{N} = (A+S_L)/(A+H)$ and $C_{\bar{L}}(\bar{N}) = (A+U_L)/(A+H)$. But $A+M+S_L = A+U\cap S+S_L = A+((U+S_L)\cap S) = A+S = L$. Since U complements C/U_L, we have that $U\cap(U_L+S_L) = U_L$, so $U\cap S_L = U_L\cap S_L = H$ and $M\cap(A+S_L) = A+(U\cap S\cap(A+S_L)) = A+U\cap S_L = A+H$. Similarly $L = A+M+U_L$ and $M\cap(A+U_L) = A+H$. It follows that the maximal subalgebra $\bar{M} = M/(A+H)$ of $\bar{L}$ complements $\bar{N}$ and $C_{\bar{L}}(\bar{N})$, and thus $\bar{L}$ is a primitive Lie algebra of type 3. Hence $M_L = A+H$.

Finally note that $M\cap U = (A+U\cap S)\cap U = A\cap U+U\cap S = B+U\cap S = U\cap S$. Similarly $M\cap S = U\cap S$.

□

The following result is straightforward to check.

Theorem 2.3 *Suppose that $L = B+U$, where B is an ideal of L and U is a subalgebra of L. Then $L/B \cong U/B\cap U$ and the following hold.*

(i) If

$$B = B_n < \ldots < B_0 = L \tag{1}$$

is part of a chief series of L, then

$$B\cap U = B_n\cap U < \ldots < B_0\cap U = U \tag{2}$$

is part of a chief series of U. If M is a maximal subalgebra of L which supplements a chief factor B_i/B_{i+1} in (1), then $M\cap U$ is a maximal subalgebra of U which supplements the chief factor $B_i\cap U/B_{i+1}\cap U$ in (2). Moreover, $(M\cap U)_U = M_L\cap U$.

(ii) Conversely, if

$$B\cap U = U_n < \ldots < U_0 = U \tag{3}$$

is part of a chief series of U, then

$$B = B+U_n < \ldots < B+U_0 = B+U = L \tag{4}$$

is part of a chief series of L. If T is a maximal subalgebra of U which supplements a chief factor U_i/U_{i+1} in (3), then $B+T$ is a maximal subalgebra of L which supplements the chief factor $(B+U_i)/(B+U_{i+1})$ in (4). Moreover, $(B+T)_L = B+T_U$.

Lemma 2.4 *Let K and H be ideals of a Lie algebra L and let*

$$K = Y_0 \subset Y_1 \subset \ldots \subset Y_{m-1} \subset Y_m = H$$

be part of a chief series of L between K and H. Suppose that A/B is a chief factor of L between K and H. Then

(i) if $A + Y_j = B + Y_j$, then $A + Y_k = B + Y_k$ for $j \leq k \leq m$;

(ii) if $A \cap Y_{j-1} = B \cap Y_{j-1}$, then $A \cap Y_{k-1} = B \cap Y_{k-1}$ for $1 \leq k \leq j$;

(iii) if $B + Y_{j-1} \subset A + Y_{j-1}$, then $B + Y_{k-1} \subset A + Y_{k-1}$ for $1 \leq k \leq j$ and $A \cap Y_{j-1} = B \cap Y_{j-1}$. In this case,

$$\frac{A + Y_{j-1}}{B + Y_{j-1}} \searrow \frac{A + Y_{k-1}}{B + Y_{k-1}} \searrow \frac{A}{B}.$$

(iv) If $B \cap Y_j \subset A \cap Y_j$, then $B \cap Y_k \subset A \cap Y_k$ for $j \leq k \leq m$ and $A + Y_j = B + Y_j$. Moreover,

$$\frac{A}{B} \searrow \frac{A \cap Y_k}{B \cap Y_k} \searrow \frac{A \cap Y_j}{B \cap Y_j}.$$

Proof.

(i) This is clear.

(ii) This is just the dual of (i).

(iii) The first assertion follows from (i). Now

$(A + Y_{k-1}) + (B + Y_{j-1}) = A + Y_{j-1}$ and $A + Y_{k-1} = B + (A + Y_{k-1})$.

Moreover, $B \subseteq B + A \cap Y_{j-1} = A \cap (B + Y_{j-1}) \subseteq A$. Since A/B is a chief factor of L, we have either $B = B + A \cap Y_{j-1} = A \cap (B + Y_{j-1})$ or $A \cap (B + Y_{j-1}) = A$. If the latter holds then $A \subseteq B + Y_{j-1}$, which implies that $A + Y_{j-1} = B + Y_{j-1}$, a contradiction. Hence $A \cap Y_{j-1} \subseteq B$, and so $A \cap Y_{j-1} = B \cap Y_{j-1}$. But now $A \cap Y_{k-1} = B \cap Y_{k-1}$, by (ii). Thus

$$A \cap (B + Y_{k-1}) = B + A \cap Y_{k-1} = B + B \cap Y_{k-1} = B,$$

and

$$\begin{aligned}(B + Y_{j-1}) \cap (A + Y_{k-1}) &= (B + Y_{j-1}) \cap A + Y_{k-1} \\ &= B + Y_{j-1} \cap A + Y_{k-1} \\ &= B + B \cap Y_{j-1} + Y_{k-1} = B + Y_{k-1},\end{aligned}$$

which completes the proof.

(iv) This is the dual of (iii).

□

Let A/B and C/D be chief factors of L such that $A/B \searrow C/D$. If A/B is a Frattini chief factor and C/D is supplemented by a maximal subalgebra of L, then we call this situation an *m-crossing*, and denote it by $[A/B \searrow C/D]$.

Note that if $[A/B \searrow C/D]$ is an m-crossing then C/D must be abelian. For, if C/D is a supplemented nonabelian chief factor, then it has a monolithic supplement, by [4, Proposition 2.5], and so A/B must also be supplemented, by Lemma 2.1 (iv).

Theorem 2.5 *Let A/C, C/D and B/D be chief factors of L. If $[A/B \searrow C/D]$ is an m-crossing, then so is $[A/C \searrow B/D]$. Moreover, in this case a maximal subalgebra M supplements C/D if and only if M supplements B/D.*

Proof. Without loss of generality we can assume that $D = 0$. Suppose that B and C are minimal ideals of L, A/B is a Frattini chief factor and C is supplemented by a maximal subalgebra M of L. Then we show that A/C is a Frattini chief factor of L and B is supplemented by M.

If $B \subseteq M$, then $L = A + M$ and $B \subseteq A \cap M$, so M supplements A/B in L, a contradiction. Hence $B \not\subseteq M$ and M supplements B.

Suppose that K is a maximal subalgebra of L that supplements A/C in L, so $L = A+K$ and $C \subseteq A \cap K$. Then $L = A+K = B+C+K = B+K$, so K also supplements B in L. Since $C \not\subseteq M_L$ and $C \subseteq K_L$, there is a maximal subalgebra $J = B + M \cap K$ such that $J_L = B + M_L \cap K_L$, by Lemma 2.2. If $A \subseteq J$ then $A = A \cap J_L = B + M_L \cap K_L \cap A = B + M_L \cap C = B$, which is a contradiction. Hence J supplements A/B. But A/B is a Frattini chief factor of L, so this is not possible. It follows that A/C is a Frattini chief factor of L. □

Proposition 2.6 *With the same hypotheses as in Lemma 2.4 assume that A/B is a supplemented chief factor of L. Let*

$$j' = max\{j : (A + Y_{j-1})/(B + Y_{j-1}) \text{ is a supplemented chief factor of } L\}$$

and put $X = Y_{j'}$ and $Y = Y_{j'-1}$. Then X/Y is a supplemented chief factor in L. Furthermore the following conditions are satisfied.

(i) If $A + X = B + X$, then $A + X = A + Y$ and

$$\frac{A}{B} \swarrow \frac{A+X}{B+Y} \searrow \frac{X}{Y}.$$

Moreover, $A \cap Y = B \cap Y = B \cap X$ and

$$\frac{A}{B} \searrow \frac{A \cap X}{B \cap Y} \swarrow \frac{X}{Y}.$$

(ii) If $A + X \neq B + X$ then

$$\left[\frac{A+X}{B+X} \searrow \frac{A+Y}{B+Y}\right]$$

is an m-crossing and

$$\frac{A}{B} \swarrow \frac{A+Y}{B+Y} \text{ and } \frac{B+X}{B+Y} \searrow \frac{X}{Y}.$$

In particular, in both cases, $(A + Y)/(B + Y)$ and $(B + X)/(B + Y)$ are supplemented chief factors of L.

Proof. Note first that $(A + Y_0)/(B + Y_0) = A/B$ is a supplemented chief factor of L, so j' is well-defined.

Suppose that $B+X = B+Y$. Then $A+X = A+Y$, so $(A+X)/(B+X) = (A + Y)/(B + Y)$ is supplemented, contradicting the choice of j'. Hence $(B + X)/(B + Y) \searrow X/Y$ and $(B + X)/(B + Y)$ is a chief factor of L.

(i) Suppose that $A+X = B+X$. Then $B+Y \subset A+Y \subseteq A+X = B+X$, so $A+X = A+Y$. Also $A/B \swarrow (A+X)/(B+Y) \searrow X/Y$ by Lemma 2.4 (iii).

Moreover, $A = A \cap (B + X) = B + A \cap X$, so $A/B \searrow A \cap X/B \cap X$. But now $A \cap Y = B \cap Y = B \cap X$, by Lemma 2.4 (iii). Hence $A/B \searrow A \cap X/B \cap Y \swarrow X/Y$.

In this case
$$\frac{B+X}{B+Y} = \frac{A+X}{B+Y} = \frac{A+Y}{B+Y}$$
is supplemented, by the definition of j'.

(ii) Suppose now that $A+X \neq B+X$. Then $(A+X)/(B+X)$ is a Frattini chief factor of L, by the choice of j'. Now $B+Y \subseteq (B+X)\cap(A+Y) \subseteq A + Y$. If $(B + X) \cap (A + Y) = A + Y$, then $A + Y \subseteq B + X$ so $A + X = B + X$, a contradiction. Hence $B + Y = (B + X) \cap (A + Y)$ and $[(A+X)/(B+X) \searrow (A+Y)/(B+Y)]$ is an m-crossing. Moreover, $A \cap Y = B \cap Y = B \cap X$, by Lemma 2.4 (iii) again, and so we have $A/B \swarrow (A + Y)/(B + Y)$ and $(B + X)/(B + Y) \searrow X/Y$.

Since $[(A+X)/(B+X) \searrow (A+Y)/(B+Y)]$ is an m-crossing, it follows from Theorem 1.1 that $(A+Y)/(B+Y)$ and $(B+X)/(B+Y)$ are supplemented chief factors of L.

In either case, $(B+X)/(B+Y)$ is a supplemented chief factor of L and $(B+X)/(B+Y) \searrow X/Y$, so X/Y is a supplemented chief factor of L. □

Proposition 2.7 *With the same hypotheses as in Lemma 2.4 assume that A/B is a Frattini chief factor of L. Let*

$$j' = max\{j : A \cap Y_j / B \cap Y_j \text{ is a Frattini chief factor of } L\}$$

and put $X = Y_{j'}$ and $Y = Y_{j'-1}$. Then X/Y is a Frattini chief factor of L. Furthermore the following conditions are satisfied.

(i) If $A \cap Y = B \cap Y$, then $A \cap Y = B \cap X$ and

$$\frac{A}{B} \searrow \frac{A \cap X}{B \cap Y} \swarrow \frac{X}{Y}.$$

Moreover, $A + Y = A + X = B + X$ and

$$\frac{A}{B} \swarrow \frac{A + X}{B + Y} \searrow \frac{X}{Y}.$$

(ii) If $A \cap Y \neq B \cap Y$ then

$$\left[\frac{A \cap X}{B \cap X} \searrow \frac{A \cap Y}{B \cap Y}\right]$$

is a crossing and

$$\frac{A}{B} \searrow \frac{A \cap X}{B \cap X} \text{ and } \frac{A \cap X}{A \cap Y} \swarrow \frac{X}{Y}.$$

In particular, in both cases, $(A \cap X)/(B \cap Y)$ and $(A \cap X)/(B \cap X)$ are Frattini chief factors of L.

Proof. This is simply the dual of Proposition 2.6. □

We say that two chief factors A/B and C/D of L are *m-related* if one of the following holds.

1. There is a supplemented chief factor R/S such that $A/B \swarrow R/S \searrow C/D$.

2. There is an m-crossing $[U/V \searrow W/X]$ such that $A/B \swarrow V/X$ and $W/X \searrow C/D$.

3. There is a Frattini chief factor Y/Z such that $A/B \searrow Y/Z \swarrow C/D$.

4. There is an m-crossing $[U/V \searrow W/X]$ such that $A/B \searrow U/V$ and $U/W \swarrow C/D$.

Theorem 2.8 *Suppose that A/B and C/D are m-related chief factors of L. Then*

(i) A/B and C/D are L-connected;

(ii) A/B is Frattini if and only if C/D is Frattini; and

(iii) if A/B and C/D are supplemented, then there exists a common supplement.

Proof.

(i) In case 1 we have

$$\frac{A}{B} = \frac{A}{A \cap S} \cong_L \frac{A+S}{S} = \frac{C+S}{S} \cong_L \frac{C}{C \cap S} = \frac{C}{D}.$$

In case 3 we have

$$\frac{A}{B} = \frac{B+Y}{B} \cong_L \frac{Y}{B \cap Y} = \frac{Y}{Z} = \frac{Y}{D \cap Y} \cong_L \frac{D+Y}{Y} = \frac{C}{D}.$$

Consider case 2. Here V/X and W/X have a common supplement, M say, by Theorem 1.1. Then $(V + M_L)/M_L$ and $(W + M_L)/M_L$ are minimal ideals of the primitive Lie algebra L/M_L. If $V + M_L = W + M_L$ then $V/X \cong_L W/X$, which implies that $A/B \cong_L C/D$. Otherwise L/M_L is a primitive Lie algebra of type 3 whose minimal ideals are $(V+M_L)/M_L$ and $(W+M_L)/M_L$. Since $A/B \cong_L V/X$ and $C/D \cong_L W/X$ we see that A/B and C/D are L-connected.

Case 4 is similar to case 2.

(ii) If A/B is Frattini, then case 1 of the definition of 'm-related' cannot hold. Suppose we are in case 2. Then $[U/W \searrow V/X]$ is an m-crossing, by Theorem 1.1, so V/X is supplemented in L. Hence A/B is supplemented in L, by Lemma 2.1, so case 2 cannot hold. If case 3 holds, then C/D is Frattini, by Lemma 2.1 (i). In case 4, $[U/W \searrow V/X]$ is an m-crossing, by Theorem 2.5. But then U/W is Frattini, whence C/D is Frattini, by Lemma 2.1.

(iii) Let A/B and C/D be supplemented. Then we are in either case 1 or case 2 of the definition of 'm-related'. In case 1, if M supplements R/S then M supplements both A/B and C/D, by Lemma 2.1. So suppose that case 2 holds, Then there is a common supplement M to V/X and W/X, by Theorem 2.5. But M also supplements A/B and C/D, by Lemma 2.1

□

Lemma 2.9 *With the same hypotheses as in Lemma 2.4 suppose that A/B and Y_j/Y_{j-1} are m-related. Then*

(i) A/B and Y_j/Y_{j-1} are supplemented in L if and only if $(A+Y_{j-1})/(B+Y_{j-1})$ is supplemented in L; and

(ii) A/B and Y_j/Y_{j-1} are Frattini in L if and only if $A \cap Y_j/B \cap Y_j$ is Frattini in L.

Proof.

(i) Put $C = Y_j$, $D = Y_{j-1}$ and suppose that case 1 of the definition of 'm-related' holds. If $A+D = B+D$ then $R = A+D+S = B+D+S = S$, a contradiction, so $B+D \subset A+D$. It follows from Lemma 2.4 (iii) that $(A+D)/(B+D) \searrow A/B$, and, in particular, that $(A+D)/(B+D)$ is a chief factor of L. But $B+D \subseteq (A+D)\cap S \subseteq A+D$. Since $A+D \not\subseteq S$ we have that $(A+D)\cap S = B+D$ and $R/S \searrow (A+D)/(B+D)$. Hence $(A+Y_{j-1})/(B+Y_{j-1})$ is supplemented in L.

Now suppose that case 2 holds. If $A+D = B+D$, then $V = A+X = A+D+X = B+D+X = X$, a contradiction, so $B+D \subset A+D$ and, as above, $(A+D)/(B+D) \searrow A/B$. Now $V = A+D+X$ and $(A+D)\cap X = A\cap X+D = B+D$, so $V/X \searrow (A+D)/(B+D)$. Hence $(A+Y_{j-1})/(B+Y_{j-1})$ is supplemented in L.

The converse follows from Lemma 2.4 (iii).

(ii) This the dual statement to (i).

□

3 A generalised Jordan-Hölder Theorem

Theorem 3.1 *Let K and H be ideals of L such that $K \subset H$ and two sections of chief series of L between K and H are*

$$K = X_0 \subset X_1 \subset \ldots \subset X_n = H$$

and

$$K = Y_0 \subset Y_1 \subset \ldots \subset Y_m = H.$$

Then $n = m$ and there is a unique permutation $\sigma \in S_n$ such that X_i/X_{i-1} and $Y_{\sigma(i)}/Y_{\sigma(i)-1}$ are m-related, for $1 \leq i \leq n$. Furthermore

$$\sigma(i) = max\{j : (X_i + Y_{j-1})/(X_{i-1} + Y_{j-1}) \text{ is supplemented in } L\}$$

if X_i/X_{i-1} is supplemented in L, and

$$\sigma(i) = min\{j : X_i \cap Y_j/X_{i-1} \cap Y_j \text{ is Frattini in } L\}$$

if X_i/X_{i-1} is Frattini in L

Proof. We can assume without loss of generality that $n \geq m$. Put $A = X_i$, $B = X_{i-1}$, $X = Y_{\sigma(i)}$, $Y = Y_{\sigma(i)-1}$.

By Proposition 2.6, if A/B is supplemented in L, then so is X/Y. Moreover, if $A + X = B + X$ then $A/B \swarrow (A + Y)/(B + Y) \searrow X/Y$, by Proposition 2.6 (i). Also $(A + Y)/(B + Y)$ is supplemented in L, by the definition of $\sigma(i)$. Thus, this is case 1 of the definition of 'm-related'. If $A + X \neq B + X$ then we are in case 2 of the definition, by Proposition 2.6 (ii).

Dually, by Proposition 2.7, if A/B is Frattini, then so is X/Y. Moreover, if $A \cap X = B \cap X$, then $A/B \searrow A \cap X/B \cap Y \swarrow X/Y$, by Proposition 2.7 (i). Also $A \cap X/A \cap Y$ is Frattini, by the definition of $\sigma(i)$. Thus, this is case 3 of the definition of 'm-related'. If $A \cap X \neq B \cap X$ then we are in case 4 of the definition, by Proposition 2.7 (ii).

Therefore, in all cases, A/B and X/Y are m-related for $1 \leq i \leq n$.

Next we show that the map $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$ defined in the statement of the theorem is injective. Put $C = X_k$ and $D = X_{k-1}$, where $i < k$ and $\sigma(i) = \sigma(k)$.

Suppose that A/B is supplemented in L; then so are X/Y and C/D. Suppose that $A+X = B+X$. Then $A \subseteq D$, so $A+X = A+Y$, by Proposition 2.6 (i), which yields that $D + X = D + A + X = D + A + Y = D + Y$. Since C/D is supplemented in L and $\sigma(k) = j$, $(C+Y)/(D+Y)$ is a chief factor of L, and then $D + X = D + Y \subset C + Y = C + X$. It follows from Proposition 2.6 (ii) that $(D + X)/(D + Y) \searrow X/Y$; in particular, $D + X \neq D + Y$, a contradiction.

Hence $B + X \subset A + X$. Then $[(A + X)/(B + X) \searrow (A + Y)/(B + Y)]$ is an m-crossing, by Proposition 2.6 (ii), and hence so is $[(A + X)/(A + Y) \searrow (B + X)/(B + Y)]$, by Theorem 2.5. It follows that the chief factor $(A + X)/(A + Y)$ is Frattini. Since $\sigma(k) = j$ we have that $(C + Y)/(D + Y)$

and $(D+X)/(D+Y)$ are supplemented chief factors of L. But $A \subseteq D$ so $A+Y \subseteq D+Y$ and $A+X \subseteq D+X$. Also $D+X = (D+Y)+(A+X)$ and $(D+Y)\cap(A+X) = A+D\cap X+Y$. But $Y \subseteq D\cap X+Y \subseteq X$ and X/Y is a chief factor of L. If $D\cap X+Y = X$ then $X \subseteq D+Y$ and so $D+X = D+Y$, contradicting the fact that $(D+X)/(D+Y)$ is a chief factor of L. Hence $D\cap X+Y = Y$, giving $D\cap X \subseteq Y$ and $(D+Y)\cap(A+X) = A+Y$. Thus $(D+X)/(D+Y) \searrow (A+X)/(A+Y)$, which implies that $(D+X)/(D+Y)$ is Frattini, by Lemma 2.1, which is a contradiction

We have shown that the restriction of σ to the subset $\mathcal{I}$ of $\{1,\ldots,n\}$ composed of all indices i corresponding to the supplemented chief factors X_i/X_{i-1} is injective. Applying dual arguments shows that the restriction of σ to the subset of $\{1,\ldots,n\}\setminus\mathcal{I}$ consisting of all Frattini chief factors X_i/X_{i-1} is injective. By arguments at the beginning of the proof, σ is injective. Hence $n=m$ and $\sigma \in S_n$.

Finally, if τ is any permutation with the above properties then the definition of σ requires $\tau(i) = \sigma(i)$ for all $i \in \mathcal{I}$ and $\tau(i) = \sigma(i)$ for all $i \in \{1,\ldots,n\}\setminus\mathcal{I}$, by Lemma 2.9. Hence $\tau = \sigma$. □

Corollary 3.2 *Let σ be the permutation constructed in Theorem 3.1. If X_i/X_{i-1} and $Y_{\sigma(i)}/Y_{\sigma(i)-1}$ are supplemented, then they have a common supplement. Moreover, the same is true if we replace 'supplement' by 'complement'.*

Proof. The first assertion follows immediately from Theorem 2.8. The second is clear if both chief factors are abelian. So suppose that they are complemented nonabelian chief factors. Then case 2 of the definition of m-related cannot hold, since W/X (and thus $Y_{\sigma(i)}/Y_{\sigma(i)-1}$) would have to be abelian, by the remark immediately preceding Theorem 2.5. Case 3 cannot hold, since Y/Z is not supplemented, and case 4 cannot hold, since U/W is not supplemented. Hence case 1 holds, and there is a supplemented chief factor R/S such that $X_i/X_{i-1} \swarrow R/S \searrow Y_{\sigma(i)}/Y_{\sigma(i)-1}$, so $R = X_i + S = Y_{\sigma(i)} + S$, $X_i \cap S = X_{i-1}$ and $Y_{\sigma(i)} \cap S = Y_{\sigma(i)-1}$.

Let M be a complement of X_i/X_{i-1}, so $L = X_i + M$ and $X_i \cap M = X_{i-1}$. Moreover, it is also a supplement of R/S and $Y_{\sigma(i)}/Y_{\sigma(i)-1}$, by Lemma 2.1 (iv), so $L = R+M = Y_{\sigma(i)}+M$, $S \subseteq R\cap M$ and $Y_{\sigma(i)-1} \subseteq Y_{\sigma(i)} \cap M$. Then

$$R + M_L = X_i + S + M_L = X_i + M_L = Y_{\sigma(i)} + M_L$$

and

$$M \cap (Y_{\sigma(i)} + M_L) = M \cap (X_i + M_L) = X_{i-1} + M_L = M_L.$$

Hence $M \cap Y_{\sigma(i)} = M_L \cap Y_{\sigma(i)} = Y_{\sigma(i)-1}$ and M complements $Y_{\sigma(i)}/Y_{\sigma(i)-1}$. □

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